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Pseudo-contractive mappings and fixed points and it's application in social, economical, science and different fields

* **Chetan Kumar Sahu**
Subhash Chandra Shrivastava
***S. Biswas**

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We give some Reviews on Pseudo contractive mapping & fixed point Theory .We also show how the fixed point theory is applicable in different stream. We reviews the proof of some fixed point theorems for pseudo-contractive mappings in Banach space and different spaces.

Keywords : Fixed points, contraction mapping, pseudo contractive mapping non-expansive mapping, multifunction, commuting mappings.

INTRODUCTION

Most important nonlinear problems of applied mathematics reduce to finding resolutions of nonlinear functional equations (e.g. nonlinear integral equations, boundary value problems for nonlinear ordinary or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations). It can be formulated in terms of finding the fixed points of a given nonlinear mapping on an infinite dimensional function space X into itself.

The theory of fixed point is one of the most powerful tool of modern mathematical analysis. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology & geometry which has many applications in various fields such as mathematics engineering, physics, economics, game theory, biology, chemistry, optimization theory and approximation theory etc. Fixed point theory has its own importance and developed tremendously for the last one and half century. The purpose of the present paper is to study the development of fixed point theory

Origin of pseudo contractive mapping

The definition of monotone operator intimately related to pseudo contractive mapping was first given by Kachurovski and iterative methods for strongly monotone operators in Hilbert space satisfying a Lipschitz condition were first given by Zarantonello and Vainberg . The first surjectivity theorem for monotone operators was given by Minty. Later, Surjectivity results proved for different type of monotone operators were applied to get existence and uniqueness results for corresponding operator equations. This Theory is now widely developed

* *Kalinga Universitys Rauipr*
***Rungta College of Engineering & Technology Bhilai*
****Kalinga Universitys Rauipr*

The notion of monotone operators could be considered as a generalization of the concept of accretive operators in the Hilbert space H i.e. F : H→H is monotone if and only if (J + λ F) is accretive for every λ > 0. The accretive operators were introduced independently in 1967 by Browder and Kato. Interest in such mappings developed mainly due to their connection with equations of evolution. It is well known that many physically significant problems can be modeled by initial value problems of the form x'(t) + Ax(t) = 0, x(O) = X_o, where A is an accretive operator in an appropriate Banach space. Typical examples where such evolutions occur can be found in the heat, wave or schrodinger equations. The solutions of the equation Ax = 0 are precisely the equilibrium points of the system . The class of non expansive mappings in Banach spaces has very interesting links with the theory of monotone and accretive mappings. The existence of fixed points of non expansive mapping was independently given by Browder , Gohde and Kirk in 1965. According to them non expansive, accretive and monotone mappings are related to each other as follows:
If T is a non expansive mapping, then U : I-T is monotone in Hilbert space from any subset D of H into H and accretive operator in Banach space into itself. But converse of this is not valid i.e. if U is monotone or accretive operator then T = I -U is not non expansive . This is the reason why pseudo contractive operator is introduced. The class of pseudo contractive operator is introduced by Browder and Petryshyn in 1967 in Hilbert space and proved that U is pseudo contractive operator if and only if T = I - U is monotone operator. They proved the existence results and then convergence results for this class of mappings in Hilbert space using Krasnoselskij iteration. In the same year, Browder alone gave the existence of fixed points of pseudo contractive mapping in real uniformly convex Banach space or real Banach space with uniform structure and proved that the class of pseudo contractive operators includes the important class of non expansive operators and also showed that T is pseudo contractive if and only if
A = I - T is accretive. Fixed point theorems for pseudo contractive mappings play an important role in the theory of nonlinear mappings because of their connection with the accretive operators .Browder and Kato independently of each other, characterized pseudo contractive mappings as those mappings T for which the mapping A = I - T is accretive. Consequently, considerable efforts have been devoted to the methods of approximating equilibrium points

(when they exist) of the initial value problems . Since a map T is pseudo contractive if and only if A = I-T is accretive so that solution of Ax = 0 for accretive operator A corresponds to fixed point of T.

HISTORY OF FIXED POINT THEORY

In the 19th century The study of fixed point theory was initiated by Poincare and in 20th century developed by many mathematicians like Brouwer, Schauder, Kakutani, Banach, Kannan ,Tarski , and others:

Let T be a self mapping on a set X. An element u in X is said to be a fixed point of the mapping T if Tu = u. The fpth (= fixed point theorem) is a statement which
asserts that under certain conditions (on the mapping T and on the space X), a mapping T of X into itself admits one or more fixed points. History is a meaningful
record of man's achievement and historical research is the application of scientific
method to the description and analysis of past events. In this paper, we have tried
briefly to present a history of pseudo contractive mapping & fixed point theorem. There are plenty of results on different cases of fixed point theorems and this paper is basically a survey work, which deals with almost all earlier settings of fixed point theorems, with suitable examples. Historically, the most important result in the fixed point theorems is the famous theorem of L.E.T. Brouwer which says that every continuous self-mapping of the closed unit in R^n, the n-dimensional Euclidean space, possesses a fixed point. This result, published by Brouwer (1910), was previously known to H. Poincare in an equivalent form.

In 1986, poin proved the following result the following result: If $f:E_n \rightarrow E_n$ is any continuous function with the property that, for some r > 0 and say a > 0,
$f(x)+a(x)\neq o$, $\|x\| = r$ then there exists a point x_0 ; $\|x_0\| \leq r$ such that $f(x_0)= x_0$.

A.L.Cauchy (1844) was the first mathematician to give a proof for the existence and
uniqueness of the solution of the differential equations $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$ when f is a continuous differentiable function. R. Lipschitz (1877) simplified Cauchy's proof using which is known today as 'Lipschitzs' condition . later G. Peano (1890) established the deeper result, supposing only the continuity of F. Peano's approach is more related to modern fixed point theorems, which is used l to obtain existence theorem.

Also, E. Sperner (1928) proved the combinatorial geometric lemma on the decomposition of a triangle, which plays an important role in the theory of fixed points.These are the most important tool, for proving the existence and uniqueness of solutions to various mathematical modols (differential, integral,ordinary and partial differential

equations, variational inequalities), other fields are
Steady-state tempreture distribution,Chemical reactions, Neutron transport theoty, Economic theory, economic theory, game theory, Flow of fluids' Optimal control theory, Fractals, etc.

SOME DEFINATION

1. **metric space:**A metric space is a set *X* together with a function *d* (called a **metric** or "distance function") which assigns a real number *d*(*x*, *y*) to every pair *x*, *y* ∈ *X* satisfying the properties (or *axioms*):

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

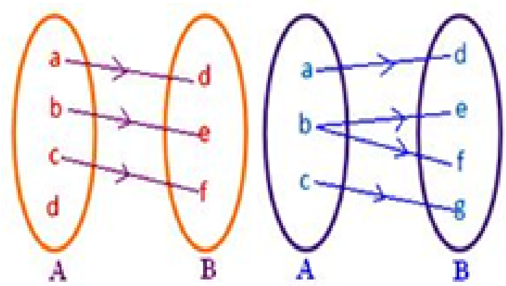
2. Banach Space: A normed space X is called a Banach space if it is complete, i.e.,if every Cauchy sequence is convergent.That is, $\{f_n\}$ n ∈ N is Cauchy in X then ∃!f ∈X such that $f_n \to f$

3.Hilbert Space: A complex inner product space together with inner product :a function from X×X→C satisfying:
(1) (x y) ≥0, (x y)=0 iff x=0 for all x, y in X
(2) (ax+ay,z) =a(x,z) + b(y,z) ∀′a, b in C and ∀′x, y, z in X
(3) $(x,y) = \overline{(y, x)}$

4.**Mapping or Functions***:* If A and B are two non-empty sets, then a relation 'f' from set A to set B is said to be a function or mapping,

- If every element of set A is associated with unique element of set B.
- The function 'f' from A to B is denoted by f : A→ B

? If f is a function from A to B and x ∈ A, then f(x) ∈ B where f(x) is called the image of x under f and x is called the pre image of f(x) under 'f'.



5.**contraction mapping:** Let X be a complete metric space. Then a map T: X→ X is called a contraction mapping on X if there exists k ∈ (01) such that
d(T(x),T(y))=d(x y) for all x and y in X

6.Non Expansive & Pseodo Contractive Mapping: Let E be a Banach space, X a subset of E, and f a mapping of X into E. Then f is said to be non expansive if for all x, y ∈ X, ||f(x)-f(x)|| ≤ ||x-y|| while f is said to be pseudo-contractive if for all x, y ∈ X and r > 0, ||x-y|| ≤ ||(1+r)(x-y)-r(f(x)-f(y)||

7. **Lipschitzian mapping**: Let (M,ρ) be a metric space and let T : M→ M be a mapping. We say that T satisfies a Lipschitz condition with constant k = 0 if for all x, y ∈ M,
ρ(T x, T y) = kρ(x, y) then T is called Lipschitzian mapping and k is called Lipshhitzian constant.

8. 2-Normed Space: Let X be a real vector space of dimension d, where $2 \leq d$. A real-valued function $\|.,.\|$ on X_2 satisfying the following four conditions:
(1) $\|x_1,x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,
(2) $\|x_1,x_2\|$ is invariant under permutation,
(3) $\|\alpha x_1,x_2\| = |\alpha| \|x_1,x_2\|$, for any $\alpha \in R$,
(4) $\|x+x',x_2\| \leq \|x,x_2\| + \|x',x_2\|$, is called a 2-norm on X, and the pair $(X, \|.,.\|)$ is called a 2-normed space.

Some Theorems

1: Let X be a Banach space, $K \subseteq X$ a closed convex subset, $U \subseteq K$ a bounded set, open in K and $u_0 \in U$ a fixed element. Assume that the operator $T : \overline{U} \to K$ is completely continuous and satisfies the boundary condition $u \neq (1 - \lambda) u_0 + \lambda T(u)$ for all $u \in \partial U, \lambda \in (0,1)$ Then T has at least one fixed point in $\overline{U}$.

2: Let X be a subset of a Banach space E and let $f: X \to E$ be a continuous pseudo-contractive mapping. If $A_f : X \to E$ is defined by $A_f : X \to E$ then:
(a) A_f is one-to-one and A_{f-1} is non expansive.
(b) f and A_f have the same fixed points.
(c) If X is closed, $A_f [X]$ is closed.
(d) If X is open, then $A_f [X]$ is open.

3: Let X be a hounded closed subset of a Banach space E (with int $(X) \neq \Phi$). Suppose $f: X \to E$ is a continuous pseudo contractive mapping and suppose there exists $z \in X$ such that $\|z-f(z)\| < \|x-f(x)\|$ for all $x \in \partial X$. Then $\inf \{ \|x - f(x)\| : x \in X\} = 0$. If in addition X has the fixed point

4: Let E be a Banach space, $f:E \to E$ a continuous pseudo-contractive mapping and suppose that for some $\partial > 0$ the set $\{x \in E: \|x \longrightarrow f(x)\| \leq \partial\}$ is nonempty and bounded. Then $\inf \{ \|x- f\{x)\| : x \in E\}$. If in addition closed balls in E have the fixed-point property with respect to non expansive self-mappings, then f has a fixed point in E.

5: Let E be a Banach space and $T: E \to E$ a continuous accretive transformation such that $\|T(x)\| \longrightarrow \infty$ as $\|x\| \to \infty$ Then the range of T is dense in E. If in addition closed balls in E have the fixed point property with respect to non expansive self-mappings, then the range of T is all of E.

6: Suppose E is a reflexive Banach space such that every nonempty closed bounded and convex subset of E has the fixed point property with respect to non expansive self mappings and suppose $f:E \longrightarrow E$ is a continuous pseudo-contractive mapping. If $x_n - f(x_n) \to 0$ strongly for some bounded sequence $\{x_n\} \subseteq E$, then f has a fixed point.

7:Let E be a uniformly convex Banach space, X a bounded closed convex subset of E with int $(X) \neq \Phi$ and G an open set containing X such that $dist(X, E \backslash G) > 0$. Suppose $f:G \longrightarrow E$ is a continuous pseudo-contractive mapping which sends bounded sets into bounded sets and satisfies for some $z \in$ int (X):
$f(x)-z \neq \lambda (x-z)$ for $x \in \partial X$, $\lambda > 1$.Then f has a fixed point in X

8: Let E be a Banach space, X a closed bounded and convex subset of E with $\text{int}(X) \neq \Phi$ and $f: X \rightarrow E$ a continuous pseudo-contractive mapping such that f[X] is bounded. Suppose there exists $z \in \text{int}(X)$ such that $f(x) - z \neq \lambda(x - z)$ for $x \in \partial X$, $\lambda > 1$.
Then
$\inf\{ \|x - f(x)\| : x \in X) = 0$.

9: Let X be a uniformly convex 2-Banach space and B be a closed sphere in X . Let U be a Lipschitzian pseudo contractive mapping from B to X such that U also maps boundary of B into B. Then U has a fixed point in B.

10: Let X be a 2-Banach space , G be an bounded subset of X with $G \in 0$ and U be a Lipschitzian pseudo contractive mapping from G to X satisfying
1. $U(x) \neq \lambda(X)$ if $x \in \partial G$
2. (I- U) G is closed Then U has a fixed point in G.

11. If X is a complete metric space and T : X → X is a contraction map, then f has a unique fixed point or T (x) = x has a unique solution.

APPLICATIONS TO FIXED POINT THEOREM

There are so many applications of fixed point theorems. Some of the applications are as follows:

1.Integral equations: These equations occur in applied mathematics, engineering and mathematical physics. They also arise as representation formulas in the solution of differential equations.

2.The Method of Successive Approximations: This method is very useful in determining solutions of integral, differential and algebraic equations.

3. Chemistry: We consider the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction. For steady-state solutions, the model can be reduced to the ordinary differential equation

$U'' - \lambda u' + F(\lambda, u, \beta, u) = 0$
$u'' - \lambda u' + F(\lambda, u, \beta, u) = 0$

where

$F(\lambda, \mu, \beta, u) = \lambda\mu(\beta - u)\exp(u)$
$F(\lambda, \mu, \beta, u) = \lambda\mu(\beta - u)\exp(u)$

(The unknown *u* represents the steady-state temperature of the reaction, and the parameters λ, μ and β represent the Peclet number, the Damkohler number and the dimensionless adiabatic temperature rise respectively. This problem has been studied by various Authors who have demonstrated numerically the existence of solutions (sometimes multiple solutions) for particular parameter ranges.

1. **Econamics**: In [9], Z.D.Mitrovic' has used some results by S.Park [11] and derived a sufficient condition for existence of an equilibrium point in

the economic model of supply and demand for finite dimensional topological vector space. to any hyper convex space.

2. **Game theory:** We consider a game with n ≥ 2 players, under the assumption that the players do not cooperate among themselves. Each players pursue a strategy, in dependence of the strategies of the other players. Denote the set of all possible strategies of the k^{th} player by K_k, and set $K = K_1 \times \cdots \times K_n$. An element $x \in K$ is called a strategy profile. For each k, let $f_k : K \to R$ be the loss function of the k^{th} player. If

$$\sum_{k=1}^{n} \mathbf{fk(x)} = 0, \quad x \in K$$

the game is said to be of zero-sum. The aim of each player is to minimize his loss, or, equivalently, to maximize his gain.

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